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NORMAL WAVES IN A FLUID-FILLED CYLINDRICAL CAVITY LOCATED IN A SATURATED POROUS MEDIUM*

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Within the framework of the linear dynamics of saturated porous media, the problem of the propagation of elastic harmonic waves in the neighbourhood of a cylindrical cavity filled with an ideal fluid and located in a porous medium is solved. The dependence of the kinematic and dynamic parameters of the normal waves on the elastic properties and permeability of the rock is found. It is shown that the permeability affects the normal wave damping in the cavity considerably.

The solution of the problem of elastic wave propagation in a cylindrical waveguide with a fluid surrounded by a permeable medium is of interest in connection with the development of an acoustic method of investigating wells, in acoustic measurements of the sound absorbing properties of porous materials, etc.

Let an infinite cylindrical cavity of radius a, filled with an ideal compressible fluid, intersect a saturated compressible medium. Let the z axis of a cylindrical r, θ, z system of coordinates be superposed on the cavity axis. A point source of harmonic vibrations of bulk velocity is placed at the origin.

Elastic wave propagation in a fluid filling a cavity is described by the equation

$$(\Delta + k_0^2) L = \frac{4\pi D}{i\omega} \delta(r, \theta, z); \quad k_0 = \frac{\omega}{c}, \quad c = \frac{1}{\sqrt{\rho_f \beta}} \tag{1}$$

where $\mathbf{W} = \nabla L$ is the displacement vector, k_0 is the wave number of the longitudinal wave in the fluid, c is the velocity of sound, β and ρ_f are the fluid compressibility and density, ω is the angular frequency, and D is the source bulk velocity.

Neglecting interphasal heat transfer and thermoeleasticity effects, the equations of small harmonic vibrations (which vary as $e^{-i\omega t}$) of a homogeneous isotropic saturated porous medium have the form /1-3/

$$\begin{aligned} &\omega^2\rho_{11}\mathbf{U} + \omega^2\rho_{12}\mathbf{V} = i\omega b(\mathbf{V} - \mathbf{U}) - N\Delta\mathbf{U} - \nabla[(A+N)\nabla\mathbf{U} + Q\nabla\mathbf{V}] \\ &\omega^2\rho_{12}\mathbf{U} + \omega^2\rho_{22}\mathbf{V} = i\omega b(\mathbf{U} - \mathbf{V}) - \nabla(Q\nabla\mathbf{U} + R\nabla\mathbf{V}) \\ &b = \eta\Phi^2/(KF(\omega)) \end{aligned} \tag{2}$$

Here $\mathbf{U}$ and $\mathbf{V}$ are the displacement vectors of the solid phase and the fluid in the pores, A, N, Q, R are elastic constants, ρ_{11}, ρ_{22} are dynamic density coefficients, $\rho_{12} < 0$ is the apparent fluid density, Φ is the bulk porosity, K is the permeability factor, and η is the fluid dynamic viscosity; the function $F(\omega)$ describes the deviation of the flow in the pores from Poiseuille flow /2/.

We represent displacement of the skeleton and the fluid in the pores as well as the displacement of the fluid filling the cavity in the form of Fourier integrals in the z-component of the wave vector

$$(\mathbf{U}, \mathbf{V}, \mathbf{W}) = \frac{1}{2\pi a} \int_{-\infty}^{\infty} d\zeta\, (\mathbf{u}, \mathbf{v}, \mathbf{w})\, e^{i\zeta z/a} \tag{3}$$

where here and henceforth we omit the factor $e^{-i\omega t}$ throughout.

We introduce the longitudinal wave potential Λ_l and transverse wave potential Ψ by the relationships

$$\begin{aligned} &\mathbf{U} = \nabla(\Lambda_1 + \Lambda_2) + \nabla \times \Psi \\ &\mathbf{V} = \nabla(m_1\Lambda_1 + m_2\Lambda_2) + \nabla \times (\Psi m_3) \\ &m_j = \frac{\gamma_{12} - \xi_j q_{12}}{\xi_j q_{22} - \gamma_{12}}, \quad k_j^2 = \xi_j^2 \frac{H}{\rho\omega} \quad (j = 1, 2), \quad m_3 = -\frac{\gamma_{12}}{\gamma_{22}} \end{aligned} \tag{4}$$

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$$\gamma_{11} = \frac{\rho_{11} + ib/\omega}{\rho}, \quad \gamma_{12} = \frac{\rho_{12} - ib/\omega}{\rho}, \quad \gamma_{22} = \frac{\rho_{22} + ib/\omega}{\rho}$$
$$q_{11} = \frac{A + 2N}{H}, \quad q_{12} = \frac{Q}{H}, \quad q_{22} = \frac{R}{H}$$
$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad H = A + 2N + 2Q + R$$

(k_j are wave numbers of longitudinal waves of the first and second kinds ($j = 1, 2$) and of the transverse waves ($j = 3$) /2/). Then (2) reduce to a system of Helmholtz vector and scalar equations /4/

$$\Delta\,(\Lambda_1, \Lambda_2, \boldsymbol{\Psi}) + (k_1\Lambda_1, k_2\Lambda_2, k_3\boldsymbol{\Psi}) = 0 \tag{5}$$

The solutions of this system and (1) that satisfy the radiation conditions and the constraints on the cavity axis have the form

$$L = \frac{1}{2\pi a}\int_{-\infty}^{\infty} d\zeta\left[X_0 J_0\left(l\frac{r}{a}\right) - \frac{D}{4\omega} H_0^{(1)}\left(l\frac{r}{a}\right)\right]e^{i\zeta z/a} \tag{6}$$
$$\Lambda_j = \frac{1}{2\pi a}\int_{-\infty}^{\infty} d\zeta X_j H_0^{(1)}\left(p_j\frac{r}{a}\right)e^{i\zeta z/a}$$
$$\boldsymbol{\Psi} = \mathbf{e}_\theta \Psi = \frac{\mathbf{e}_\theta}{2\pi a}\int_{-\infty}^{\infty} d\zeta X_3 H_1^{(1)}\left(s\frac{r}{a}\right)e^{i\zeta z/a}$$
$$p_j = \sqrt{k_j^2 a^2 - \zeta^2}, \quad s = \sqrt{k_3^2 a^2 - \zeta^2}, \quad l = \sqrt{k_0^2 a^2 - \zeta^2}$$

$H_n^{(1)}$ and J_n are the Hankel function of the first kind and the Bessel function /5/, and X_n are determined by the following conditions on the cavity boundary:

$$\Gamma_{rr} = -p_L, \quad \Gamma_{rz} = 0, \quad p_L = p_0 \tag{7}$$
$$(1 - \Phi)\,U_r^{\cdot} + \Phi V_r^{\cdot} = W_r^{\cdot}$$
$$(\Gamma_{ij} = A\delta_{ij}\,\mathrm{div}\,\mathbf{U} + 2N\,(\partial U_i/\partial x_j + \partial U_j/\partial x_i) +$$
$$Q\,\mathrm{div}\,(\mathbf{U} + \mathbf{V})\,\delta_{ij} + \delta_{ij} R\,\mathrm{div}\,\mathbf{V},$$
$$p_0 = -(Q\,\mathrm{div}\,\mathbf{U} + R\,\mathrm{div}\,\mathbf{V})/\Phi)$$

Here p_L is the pressure of the fluid filling the cavity, Γ_{ij} is the total stress tensor in the porous medium, and p_0 is the fluid pressure in the pores.

Conditions (7), respectively, express the continuity of the total normal stresses, the absence of shear stresses, the equality of the fluid pressures in the cavity and the pores, and the continuity of the normal velocity component on the cavity boundary.

Substitution of the expressions for the longitudinal and transverse wave potentials (6) into (7) results in a system of four linear equations for determining the constants X_n

$$\varkappa^2\beta^{-1}X_0 J_0(l) - \sum_{j=1}^{2} X_j[(A + Q(m_j + 1) + Rm_j)(k_j^2 a^2 + \tag{8}$$
$$2Np_j)H_0^{(1)}(p_j) - 2Np_j H_1^{(1)}(p_j)] -$$
$$2Ni\zeta X_3[sH_0^{(1)}(s) - H_1^{(1)}(s)] = -\varkappa^2\beta^{-1}DH_0^{(1)}(l)$$

$$\sum_{j=1}^{2} X_j 2i\zeta p_j H_1^{(1)}(p_j) + X_3(k_3^2 a^2 - 2\zeta^2)H_1^{(1)}(s) = 0$$
$$\Phi\varkappa^2\beta^{-1}X_0 J_0(l) - \sum_{j=1}^{2} X_j(Q + Rm_j)k_j^2 a^2 H_0^{(1)}(p_j) =$$
$$-\Phi D\varkappa^2\beta^{-1}H_0^{(1)}(l)$$
$$X_0 l J_1(l) - \sum_{j=1}^{2} X_j(1 - \Phi + m_j\Phi)\,p_j H_1^{(1)}(p_j) +$$
$$X_3(1 - \Phi + m_3\Phi)\,i\zeta H_1^{(1)}(s) = -DlH_1^{(1)}(l); \quad \varkappa = k_0 a$$

The normal waves correspond to the roots of the equation $\Delta = 0$, where Δ is determinant of system (8). This equation determines the dependence of the normal wave velocities and dampings on the frequency (dispersion relations).

The source under consideration excites two kinds of normal waves /6-8/, the Lamb-Stonely wave (LSW) and pseudo-Rayleigh waves.

Just like the zero-th normal wave propagating in a waveguide with absolutely rigid walls,

the LSW is excited for frequencies as low as desired. At the lowest frequencies the dispersion equation $\Delta = 0$ has a unique root corresponding to this wave. At high frequencies the LSW behaves just like a Stonely wave propagating along the plane interface of two media. In the low-frequency domain, when the degree of penetration into the fluid filling the cavity exceeds the cavity radius, the LSW becomes practically planar and is similar to waves considered by Zhukovskii and Lamb /9, 10/. This type of low-frequency wave is also called tubular waves, hydrowaves /10, 11/ and Stonely waves /6/ in the literature. The LSW phase and group velocities are lower than the transverse wave velocities in a solid and the longitudinal wave in a fluid filling a cavity, and the amplitude of the displacement therein falls off with distance from the interface. The properties of LSW were investigated in detail in /7, 8/ for a cavity in an ideally elastic single-phase medium.

The solution of the dispersion equation $\Delta = 0$ in the complex domain was obtained using the Powell method of searching for the minimum of a function of many variables /11/. Calculations were perfomed for a porous medium (unless otherwise stated $\Phi = 0.2$ and $K = 1\,\mu m^2$) with density $\rho_s = 2870$ kg/m^3 of the skeleton substance, with the velocities $v_p = 6800$ m/sec and $v_p/v_s = 1.8$ of the longitudinal V_p and transverse waves v_s in the skeleton substance; $a = 0.1$ m, the elastic constants A, N, Q, R were calculated by a well-known method /12/.

Fig.1 shows frequency dependences of the LSW phase velocity along the boundary with a porous permeable solid (solid curve *1*) and a single-phase elastic solid (dashed curve *1*). The computations showed that for both the single-phase and the saturated porous medium the LSW phase velocity increases with frequency.

The occurrence of filtration fluid overflows on the cavity boundary results in a noticeable decrease in the LSW velocity on the boundary with the permeable solid and the appearance of damping. Fig.1 shows the logarithmic damping decrement θ of the LSW as a function of the frequency (cures *2* and *3* correspond to $K = 1\,\mu m^2$ and $K = 0.1\,\mu m^2$). Here, the frequency dependence is represented for the normalized damping coefficient A of a LSW, which equals the abolute value of the residue of the integrand in (6) for the potential L. The solid curve *4* corresponds to a permeable medium, and the dashed curve *4* to an impermeable medium. The amplitude of the excitation factor increase monotonically as the frequency falls and depends weakly on the permeability.

Fig.2 shows the dependence of the velocity (solid curves *1* and *2*) and the logarithmic damping decrement (dashed curves *1* and *2*) on the permeability factor K over its range of variation characteristic for mountain rock. Curve *1* and *2* correspond to $\Phi = 0.1$ and $\Phi = 0.2$. As the permeability increases, the intensity of the filtration fluid overflow in the pores increases, resulting in subsequent viscous dissipation of the wave energy because of friction on the walls of the pore channels and, respectively, in an increase in the damping and a decrease in the LSW velocity. The reduction in the velocity as the porosity increases is due mainly to a decrease in the shear modulus of the rock.

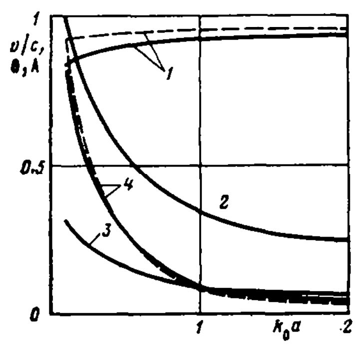

Fig.1

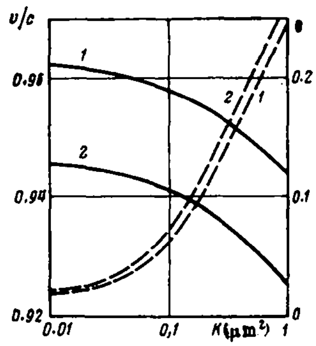

Fig.2

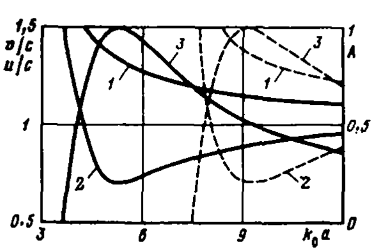

Fig.3

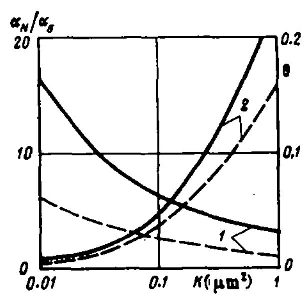

Fig.4

To obtain an explicit expression for the LSW wave number k in the limiting case of low frequencies, it is simplest to use the theory of waves in narrow tubes /10/ given in /13/. The wave number of the zero-th normal wave in a tube with walls that are not absolutely rigid is determined in the low-frequency domain $(k_0a \ll 1)$ by the expression

$$k^2 = k_0^2\left(1 - \frac{2}{ik_0ay}\right), \quad y = \frac{p}{w^{\cdot}\rho_f c} \tag{9}$$

where y is the specific acoustic impedance of the boundary and $w^{\cdot}$ is the normal component of the fluid velocity. Computations for the boundary with a porous permeable medium in the long-wave approximation $(|k_ja| \ll 1)$ results in the expression

$$y = \frac{i\beta(Q + m_2R)(k_2a)^2}{\Phi^2(k_0a)(m_2 - 1)} \ln\frac{\Gamma k_2a}{2} \tag{10}$$

$$k_2 = (1 + i)\frac{2\pi}{V_0}\left(\frac{\eta\Phi^2\omega\delta}{2(\rho_{12} + \rho_{22})K}\right)^{1/2}$$

$$v_0 = \sqrt{\frac{H}{\rho}}, \quad m_2 = \frac{q_{12} + \delta}{\delta - q_{12}}, \quad \delta = q_{11}q_{22} - q_{12}^2, \quad \Gamma = -ie^C$$

(C is Euler's constant). It follows from (9) and (10) that unlike the case of an impermeable boundary the LSW velocity tends to zero as the frequency falls. This phenomenon has a simple explanation: as the frequency falls the whole large fluid mass in the pore channels is involved in the vibrations.

At sufficiently high frequencies the dispersion equation $\Delta = 0$ has roots corresponding to the zero-th (LSW), first, second, etc. normal waves. All these waves, except the LSW, have critical velocities. In the literature devoted to normal waves in a cylindrical cavity surrounded by an impermeable elastic medium they are called reflected conical /6/, water or simply normal /7/, as well as pseudo-Rayleigh waves /8/.

The normal wave phase velocity v satisfies the inequality $c < v < v_s$ /7, 8/, where v_s is the transverse wave velocity. For a frequency close to the critical value, the phase velocity v and group velocity u of normal waves are close to the transverse wave velocity. As the frequency increases these velocities approach the longitudinal wave velocity in the fluid filling the cavity.

Fig.3 shows dispersion curves for the first normal wave (the solid curves) and the second normal wave (the dashed curves). As the frequency increases the phase velocities of the normal waves (curves *1*) decrease monotonically while the group velocities (curve *2*) depend non-monotonically on the frequency, and have a minimum. The presence of a minimum on the curve for the group velocity corresponding to the maximum of the excitation factor (curves *3*) indicates the existence of an Airy phase /7, 8/ in which the group velocity of the normal wave is below the longitudinal wave velocity in the fluid.

In the case of practical importance when the inequalities $|k_2a| \gg 1$, $\omega/\omega_0 \ll 1$ are satisfied simultaneously, where $\omega_0 = b/\rho_f$ /2/ the dispersion equation in the zero-th approximation is identical with the analogous equation for normal waves on the boundary with an ideally elastic medium.

Indeed, on replacing the Hankel functions $H_n^{(1)}(p_2)$ in (8) by their asymptotic form for $|p_2| \gg 1$ /5/, the column of the determinant Δ of system (8) containing a function of p_2 takes the form

$$\text{col}\,[2N(\zeta^2 - ip_2),\ 2\zeta p_2,\ -(Q + Rm_2)k_2^2a^2,$$
$$(1 - \Phi + m_2\Phi)p_2]\sqrt{2/(\pi p_2)}\exp[i(p_2 - \pi/4)]$$

If terms of just the highest order in k_2a are kept in the dispersion equation, taking into account that for $\omega/\omega_0 \ll 1$

$$k_1 \to \omega\sqrt{H/\rho},\ m_1 \to 1,\ k_3 \to \omega\sqrt{N/\rho},\ m_3 \to 1$$

then it is identical with the dispersion equation for a cavity located in a single-phase elastic medium with the following dynamic parameters: $\rho_0 = \rho, \lambda = H - 2N, \mu = N$. Here ρ_0 is the density, and λ, μ are the Lamé parameters. An analogous result was obtained /14/ in investigating Rayleigh wave propagation along the boundary of a saturated porous half-space by using the method of matched asymptotic expansions.

As for the LSW, the excitation factor of normal waves for a given frequency will depend slightly on the permeability and be close in magnitude to the excitation factor of these waves in a cavity located in a single-phase elastic medium. At the same time normal wave damping depends strongly on the permeability of the medium. Fig.4 shows the logarithmic damping decrement θ (curves *1*) and the ratio α_N/α_s (curves *2*), where α_N, α_s are, respectively, the normal and transverse wave damping coefficients for the first normal wave for $k_0a = 5$ (the solid curves) and for the second normal wave for $k_0a = 10$ (the dashed curves). The normal wave damping coefficient in the presence of a permeable boundary considerably exceeds the transverse wave damping coefficient.

The influence of overflows through the interface of the media on normal wave propagation reveals the possibility of an experimental determination of the filtration characteristics of a medium by means of the parameters of these waves.

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ON THE STATE OF STRESS AND STRAIN OF LAYERED PLATES OF NON-SYMMETRIC CONSTRUCTION*

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An asymptotic analysis is performed of the elasticity theory problem of the deformation of a thin multilayer **anisotropic plate** in a three-dimensional formulation without assumption regarding the regularity of the plate construction and the nature of the layer or packet deformation as a whole.

Results /1/ are used of an investigation of the solutions of elliptic boundary value problems in thin domains. The relative packet height is the small parameter h. A system of equations is obtained for the limit problem (as $h \to 0$), effective plate stiffness characteristics are found, and specific examples of their analysis are presented.

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